J. Appl. Maths Mechs, Vol. 55, No. 4, pp. 566-567, 1991
Printed in Great Britain

0021-8928/91 \$15.00+0.00

ON THE INSTABILITY OF AN EQUILIBRIUM POINT OF A LINEAR OSCILLATOR WITH VARIABLE PARAMETERS*

A.O. IGNAT'YEV

Donetsk

(Received 10 May 1990)

The conditions of instability of the position of equilibrium are obtained for a linear oscillator with variable parameters, and illustrated for the case of plane oscillations of a rocket with fins about its centre of mass.

Consider a linear oscillator

$$\ddot{x} + f(t)x^{\cdot} + g(t)x = 0 \tag{1}$$

where $f(t)$ and $g(t)$ are continuously differentiable functions of time t, with $t \geqslant 0$. The problem of its position of equilibrium

$$x = 0,\ x^{\cdot} = 0 \tag{2}$$

has been studied in a number of publications (e.g. /1, 2/). The sufficient conditions for asymptotic stability and the instability of solution (2) of Eq.(1) were studied in /3, 4/. Below we establish the sufficient conditions of instability of the position of equilibrium (2) of a linear oscillator (1), which differ from those indicated above.

Writing $x^{\cdot} = y$, we obtain a system of equations

$$x^{\cdot} = y,\ y^{\cdot} = -g(t)x - f(t)y \tag{3}$$

equivalent to Eq.(1) and having a trivial solution

$$x = 0,\ y = 0 \tag{4}$$

Theorem 1. Solution (4) of system (3) is unstable if $t_0 > 0$ exists such that when $t \geqslant t_0$ one of the following conditions holds:

$$D(t) = {}^1\!/_4 f^2(t) - g(t) < 0 \tag{5}$$

$$D(t) > 0,\ 4f(t)D(t) + {}^1\!/_2 f^{\cdot}(t) f(t) - g^{\cdot}(t) - (f^{\cdot}(t) - f^2(t) - 4D(t))\sqrt{D(t)} < 0) \tag{6}$$

Proof. Let ε be an arbitrary positive number. We shall show that for any, arbitrarily small $\delta > 0$ we can choose x_0, y_0, satisfying the inequalities

$$|x_0| < \delta,\ |y_0| < \delta \tag{7}$$

such that $T > 0$ will exist such that the trajectory $x(t),\ y(t)\ (x(t_0) = x_0,\ y(t_0) = y_0)$ will attain, at $t = t_0 + T$, the boundary of the region

$$|x| < \varepsilon,\ |y| < \varepsilon \tag{8}$$

Let us consider the function $V = xy$. Its derivative has, by virtue of (3), the form

$$V^{\cdot} = y^2 - f(t)xy - g(t)x^2$$

Let us choose $x_0 > 0,\ y_0 > 0$ satisfying conditions (7) and $V^{\cdot} > 0$, and consider the trajectory of motion $x(t),\ y(t)$ satisfying the initial conditions $x(t_0) = x_0,\ y(t_0) = y_0$. We can assume without loss of generality that $D(t_0) < 0$. Let $[t_0; t_1], [t_2; t_3], \ldots, [t_{2n}; t_{2n+1}], \ldots$ be the intervals in which condition (5) holds, and $(t_1; t_2; t_3; t_4), \ldots, (t_{2n-1}; t_{2n}) \ldots$ the intervals in which inequalities (6) hold. Since condition $V^{\cdot} \geqslant 0$ holds in the interval $[t_0; t_1]$, it follows that the trajectories in this interval will lie in the region $xy \geqslant x_0 y_0$. Let us now consider $x(t),\ y(t)$ in the interval $(t_1; t_2)$. The derivative $V^{\cdot}$ changes its sign in this interval and $V^{\cdot} = 0$ when

$$y = ({}^1\!/_2 f + \sqrt{D})x \tag{9}$$

and $V^{\cdot} > 0$ when

$$y > ({}^1\!/_2 f + \sqrt{D})x \tag{10}$$

*Prikl.Matem.Mekhan.,55,4,701-703,1991

Let $t_* \in [t_1; t_2]$ be such an instant of time that $x(t_*), y(t_*)$ satisfy the condition $y(t_*) = (^1/_2 f(t_*) + \sqrt{D(t_*)}) x(t_*)$, i.e. that a point of the trajectory lies on the straight line (9) when $t = t_*$.

We shall show that $x(t), y(t)$ when $t \in (t_*, t_* + \Delta t)$ satisfies inequality (10) provided that $\Delta t > 0$ is sufficiently small. To do this we shall calculate the second derivative of V and write its formula under the condition that $V^{\cdot} = 0$:

$$V^{\cdot\cdot}|_{(9)} = -(4f(t)D(t) + {}^1/_2 f^{\cdot}(t)f(t) + g^{\cdot}(t) + (f^{\cdot}(t) + f^2(t) + 4D(t))\sqrt{D(t)})x^2$$

Taking into account condition (6) we find that $V^{\cdot\cdot} > 0$ when $V^{\cdot} = 0$, i.e. when $t \in (t_*; t_* + \Delta t)$ the trajectory belongs to the region $V^{\cdot} > 0$. This shows that when $t \in [t_1; t_2]$ the trajectory lies in the region $V^{\cdot} \geqslant 0$.

Similary we can show that when $t \in [t_n; t_{n+1}]$ where $n = 3, 4, \ldots$, the point $x(t), y(t)$ lies in the set $V^{\cdot} \geqslant 0$. This means that the relation $V^{\cdot}(x(t), y(t)) \geqslant 0$ holds for the trajectory in question for any $t \geqslant t_0$. The latter inequality implies that when $t \geqslant t_0$, the trajectory will lie in the region $xy \geqslant x_0 y_0$.

We shall show that the boundary of the region (8) can be attained within a finite length of time. Let us consider, in the xy-plane, the region

$$\Omega = \{x, y\colon xy \geqslant x_0 y_0,\ 0 < x < \varepsilon,\ 0 < y < \varepsilon\}$$

Let us obtain an estimate of the time during which the trajectory $x(t), y(t)$ can remain in Ω. The inequality $y \geqslant x_0 y_0 \varepsilon^{-1}$ holds within this region, therefore by virtue of the first equation of (3) we have $x(t) \geqslant x_0 + x_0 y_0 \varepsilon^{-1}(t - t_0)$. From the last relation it follows that the time interval during which the trajectory may remain in Ω can be given by the number $T = \varepsilon(\varepsilon - x_0) x_0^{-1} y_0^{-1}$. Since the trajectory cannot leave the region Ω by intersecting the hyperbola $xy = x_0 y_0$, this violates one of the inequalities of (8), thus completing the proof of the theorem.

The equations of small plane oscillations of the rocket whose centre of gravity moves rectilinearly with constant velocity /5/ have the form (1), where $f(t) = ae^{-\alpha t}$, $g(t) = b^2 e^{-\alpha t}$, a, b, α are constant numbers and $\alpha > 0$. The quantity x represents, in this case, the angle of attack. The sufficient condition of the stability of solution (2) of system (1) was obtained in /5/, and it was shown that in the case of plane oscillations of the rocket the condition does not hold. We shall use the theorem given above to show that the position of equilibrium (2) is unstable. Indeed, there exists $t_0 > 0$ such, that relations (6) hold when $t \geqslant t_0$. This proves the Lyapunov instability of the small oscillations of the rocket.

Theorem 2. If the functions $f(t)$ and $g(t)$ in Eq.(1) are non-vanishing, i.e. if the following limits hold:

$$\lim_{t\to\infty} f(t) = 0, \quad \lim_{t\to\infty} g(t) = 0$$

the position of equilibrium (2) cannot be uniformly stable.

Proof. Let us consider a system of differential Eqs.(3) admitting of the trivial solution (4). Take any $\varepsilon > 0$. We shall show that for any $\delta > 0$ there exists x_0, y_0 satisfying the inequalities (7) and $t_0 \geqslant 0$ such that the trajectory $x(t), y(t)$ where $x(t_0) = x_0$, $y(t_0) = y_0$, leaves the region (8) as the time increases. Let us write

$$\sigma(t) = {}^1/_2 |f(t)| + {}^1/_2 \sqrt{f^2(t) + 4|g(t)|}$$

Since $f(t)$ and $g(t)$ vanish, so will $\sigma(t)$. We choose $t_0 > 0$ so that the inequality $\sigma(t) < x_0 y_0 \varepsilon^{-2}$ holds for $t \geqslant t_0$. Here the trajectories $x(t), y(t)$ will be situated, for $t \geqslant t_0$, in the region $V^{\cdot} > 0$. Then, as follows from the proof of Theorem 1, an instant of time $t > t_0$ will exist, in which the trajectory will leave the region (8), which it was required to prove.

REFERENCES

1. YAKUBOVICH V.A. and STARZHINSKII V.M., Parametric Resonance in Linear Systems. Nauka, Moscow, 1987.
2. STARZHINSKII V.M., Sufficient conditions of stability of a mechanical system with a single degree of freedom. PMM, 16, 3, 1952.
3. SAVCHENKO A.YA. and IGNAT'YEV A.O., Some Problems of the Stability of Non-selfsimilar Dynamic Systems. Nauk. Dumka, Kiev, 1989.
4. IGNAT'YEV A.O., On the stability of the position of equilibrium of oscillatory systems with variable coefficients. PMM, 46, 1, 1982.
5. MOISEYEV N.N., Asymptotic Methods of Non-linear Mechanics. Nauka, Moscow, 1969.

Translated by L.K.